Extended Spectrum, Extended Eigenspaces and Normal Operators

Gilles Cassier, Hasan Alkanjo

EXTENDED SPECTRUM, EXTENDED EIGENSPACES AND SELF-ADJOINT OPERATORS

GILLES CASSIER AND HASAN ALKANJO

Abstract. We say that a complex number λ is an extended eigenvalue of a bounded linear operator T on a Hilbert Space $\mathcal{H}$ if there exists a nonzero bounded linear operator X acting on $\mathcal{H}$, called extended eigenvector associated to λ, and satisfying the equation $TX = \lambda XT$. In this paper we deal the set of extended eigenvalues for the product of a positive and a self-adjoint operator which are both injective. We also describe the set of extended eigenvectors of these operators.

1. Introduction And Preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{L}(\mathcal{H})$, then a complex number λ is an extended eigenvalue of T if there is a nonzero operator X such that $TX = \lambda XT$. We denote by the symbol $\sigma_{ext}(T)$ the set of extended eigenvalues of T. The set of all extended eigenvectors corresponding to λ will be denoted as $E_{ext}(\lambda)$. Extended eigenvalues and their corresponding extended eigenvectors were studied in [1], [2] and [4].
In [2], Biswas, Lambert and Petrovic have introduced this notion and they determined the set $\sigma_{ext}(V)$ where V is the well-known integral Volterra operator on the space $L^2[0,1]$. In [4], Karaev gave a complete description of the set of extended eigenvectors of V.
In this paper we deal a large class of operators, that is the self-adjoint operators. Let T in $\mathcal{L}(\mathcal{H})$, and let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_c(T)$ denote the spectrum, the point and the continuous spectrum of T respectively. Using a theorem of Rosenblum [5], it was established in [3] that

$$\sigma_{ext}(T) \subset \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}. \tag{1.1}$$

It is known that for any self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) \subset \mathbb{R}$ and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$. Obviously, if T is a non-injective self-adjoint operator, then $\sigma_{ext}(T) = \mathbb{C}$. Indeed, for all $\lambda \in \mathbb{C}$, one can take X being a nonzero operator from kernel of T to itself. In addition, if $\sigma(T) \subset \mathbb{R}^*$ then by (1.1) $\sigma_{ext}(T) \subset \mathbb{R}$. Indeed, $\sigma(\lambda T) = \{\lambda t : t \in \sigma(T)\}$. Consequently, $\sigma(T) \cap \sigma(\lambda T) = \emptyset$ for all $\lambda \in \mathbb{C} \backslash \mathbb{R}^*$.
So, in what follows, T will denote an injective non-invertible self-adjoint operator, i.e., $0 \in \sigma_c(T)$.

First we will give some auxiliary results. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}_+$ and denote by $K := [0,1] \cup [0,\lambda]$. We will show that $\mathbb{C}[X]$, the set of polynomials, is dense in the space $L^2(K)$. To do so, we recall the following Green's theorem

Theorem 1.1. *Suppose that Γ is a finite system of curves of Jordan of class C^1, and denote by $U = Ins(\Gamma)$. Let f be a function of the class $C^\infty(U)$ and* *continuous on $\overline{U}$, then for all $z \in U$ we have :*

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{w-z}dw - \frac{1}{\pi}\iint_{U}\frac{\partial f}{\partial \bar{w}}(w)\frac{dm_2(w)}{w-z},$$

where m_2 denotes Lebesgue measure in the plane

We also need the following lemma

Lemma 1.2. *Let $\mathcal{A}$ denote the closure of $\mathbb{C}[X]$ in the space $C(K)$, i.e., $\mathcal{A} = \overline{\mathbb{C}[X]}^{C(K)}$, and let ε_1 be the identity polynomial, i.e., $\varepsilon_1(z) = z$ for all z. Then $\sigma_{\mathcal{A}}(\varepsilon_1) = K$.*

Indeed, because $\mathcal{A} \subset C(K)$ we have $K = \sigma_{C(K)}(\varepsilon_1) \subset \sigma_{\mathcal{A}}(\varepsilon_1)$. On the other hand, $\partial\sigma_{\mathcal{A}}(\varepsilon_1) \subset \partial\sigma_{C(K)}(\varepsilon_1) = K$. Now we proceed by reduction to absurd. Suppose that $w \in \sigma_{\mathcal{A}}(\varepsilon_1)\backslash K$ and consider the set $J = \{tw : t \geq 1\}$. Clearly $J \cap K = \emptyset$ and J intersects both $\sigma_{\mathcal{A}}(\varepsilon_1)$ and $(\sigma_{\mathcal{A}}(\varepsilon_1))^c$. Now, because J is connected, there is $v \in J \cap \partial\sigma_{\mathcal{A}}(\varepsilon_1) \subset K$, which is a contradiction.

Now we can show the following lemma

Lemma 1.3. *Let $f, g \in L^2[0,1]$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$. Then there exists a sequence $(p_n)_{n\in\mathbb{N}} \subset \mathbb{C}[X]$ such that for all $t \in [0,1]$*

$$p_n(t) \to f(t) \text{ and } p_n(\lambda t) \to g(t),$$

in $L^2[0,1]$.

Proof. Let $f \in L^2(K)$ be such that $f \perp \mathcal{A}$. By using Lemma (1.2), $(z-w)^{-1} \in \mathcal{A}$ for all $w \notin K$. Thus

$$\int_K \frac{\overline{f(z)}}{z-w}d\mu(z) = 0, \ \forall w \notin K,$$

where μ is the arc-length measure. Let now $\phi_\epsilon \in C^\infty(\mathbb{R}^2)$ with compact support $\overline{U} = D(0,R]$, then

$$\iint_U \frac{\partial \phi_\epsilon}{\partial \bar{w}}(w)\left(\int_K \frac{\overline{f(z)}}{w-z}d\mu(z)\right)dm_2(w) = 0,$$

by Fubini theorem we have

$$\int_K \left(\iint_U \frac{\partial \phi_\epsilon}{\partial \bar{w}}(w)\frac{dm_2(w)}{w-z}\right)\overline{f(z)}d\mu(z) = 0,$$

thus according to Theorem 1.1,

$$\int_K \phi_\epsilon(z)\overline{f(z)}d\mu(z) = 0.$$

The appropriate choosing of ϕ_ϵ follows

$$\int_a^b \overline{f(t)}dt = 0, \text{ and } \int_{\lambda a}^{\lambda b}\overline{f(z)}d\mu(z) = 0, \ \forall a, b \in [0,1],$$

thus $f = 0$.

∎

2. Product Of Self-adjoint Operators

Lemma 2.1. *Let $R \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, and let $a > ||R||$. Then for any $p \in \mathbb{C}[X]$ we have,*

$$< p(R)x, y >= p(a) < x, y > - \int_{-a}^{a} p^{'}(t) < E(]-\infty, t])x, y > dt, \ \forall x, y \in \mathcal{H},$$

where $<,>$ denotes the standard inner product in $\mathcal{H}$, and E denotes the spectral measure associated to R. In particular, if R is a positive operator, then

$$< p(R)x, y >= p(a) < x, y > - \int_{0}^{a} p^{'}(t) < E([0, t])x, y > dt, \ \forall x, y \in \mathcal{H}.$$

Proof. First, recall that the indicator function of a subset $\Omega \subset \mathbb{R}$ is defined by

$$\mathbf{1}_{\Omega}(t) = \begin{cases} 1 & \text{if } t \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

Hence

$$\int_{-a}^{a} p^{'}(t) < E(]-\infty, t])x, y > dt = \int_{\mathbb{R}} \mathbf{1}_{[-a,a]}(t) p^{'}(t) \left(\int_{-a}^{a} \mathbf{1}_{]-\infty,t]}(s) dE_{x,y}(s) \right) dt$$

by using Fubini's theorem, we have

$$\int_{-a}^{a} p^{'}(t) < E(]-\infty, t])x, y > dt = \int_{-a}^{a} \left(\int_{\mathbb{R}} \mathbf{1}_{[-a,a]}(t) \mathbf{1}_{[s,a]}(t) p^{'}(t) dt \right) dE_{x,y}(s)$$

$$= \int_{-a}^{a} \left(\int_{s}^{a} p^{'}(t) dt \right) dE_{x,y}(s) = p(a) < x, y > - < p(R)x, y > .$$

□

Theorem 2.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$, and let $T = AB$ be such that $0 \notin \sigma_p(T) \cup \sigma_p(T^*)$ then*

1. *If $A, B \geq 0$, then $\sigma_{ext}(T) \subset \mathbb{R}_+^*$.*
2. *If $A \geq 0$ and $B = B^*$ then $\sigma_{ext}(T) \subset \mathbb{R}^*$.*

Proof. (1) If we define $R = \sqrt{A} B \sqrt{A}$, then one can show that for any $n \in \mathbb{N}$ $T^{n+1} = \sqrt{A} R^n \sqrt{A} B$. Hence for all polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$

$$p(T) = a_0 I + \sqrt{A} \sum_{k=1}^{n} a_k R^k \sqrt{A} B = a_0 I + \sqrt{A} (S^* p)(R) \sqrt{A} B,$$

where S^* denotes the well-known backward shift.
First note that, since T is injective, $0 \notin \sigma_{ext}(T)$.
Now suppose that $\lambda \in \mathbb{C} \backslash \mathbb{R}_+$, and let $X \in \mathcal{L}(\mathcal{H})$ satisfying the equation $TX = \lambda XT$, then $p(T)X = Xp(\lambda T)$ for all $p \in \mathbb{C}[X]$. Let $p(z) = \sum_{k=0}^{n} a_k z^k$, then for all $x, y \in \mathcal{H}$, we have

$$< p(T)Xx, y >= a_0 < Xx, y > + < (S^* p)(R) \sqrt{A} B X x, \sqrt{A} y > .$$

Using Lemma 2.1, we obtain

$$< p(T)Xx, y >= a_0 < Xx, y > + (S^* p)(a) < TXx, \sqrt{A} y >$$
$$- \int_{0}^{a} (S^* p)^{'}(t) < E([0, t]) \sqrt{A} B X x, y > dt.$$

On the other hand, and similarly

$$< Xp(\lambda T)x, y >= a_0 < Xx, y > +\lambda(S^*p)(\lambda a) < XTx, y >$$
$$-\lambda^2 \int_0^a (S^*p)'(\lambda t) < E([0,t])\sqrt{A}Bx, \sqrt{A}X^*y > dt.$$

Consequently, and since S^* is surjective in $\mathbb{C}[X]$, it follows that

$$q(a) < TXx, y > -\lambda q(\lambda a) < XTx, y >$$
$$= \int_0^a q'(t) < E([0,t])\sqrt{A}BXx, \sqrt{A}y > dt$$
$$-\lambda^2 \int_0^a q'(\lambda t) < E([0,t])\sqrt{A}Bx, \sqrt{A}X^*y > dt,$$

for all $q \in \mathbb{C}[X]$. Next, let $p \in \mathbb{C}[X]$ and consider $q(x) = \int_0^x p(t)dt$, we obtain that

$$\int_0^a p(t)dt < TXx, y > -\lambda^2 \int_0^a p(\lambda t)dt < XTx, y >$$
$$= \int_0^a p(t) < E([0,t])\sqrt{A}BXx, \sqrt{A}y > dt$$
$$-\lambda^2 \int_0^a p(\lambda t) < E([0,t])\sqrt{A}Bx, \sqrt{A}X^*y > dt.$$

Now we consider the set $K = [0,a] \cup [0,\lambda a]$, in virtue of the Lemma 1.2, if $z \notin K$ then $(\varepsilon_1 - zid)^{-1} \in \mathcal{A}$ and hence there exists a sequence of polynomials p_n uniformly converging towards the last function. Thus

$$\int_0^a \frac{1}{t-z}dt < TXx, y > -\lambda^2 \int_0^a \frac{1}{\lambda t - z}dt < XTx, y >$$
$$= \int_0^a \frac{1}{t-z} < E([0,t])\sqrt{A}BXx, \sqrt{A}y > dt$$
$$-\lambda^2 \int_0^a \frac{1}{\lambda t - z} < E([0,t])\sqrt{A}Bx, \sqrt{A}X^*y > dt.$$

Now let $\epsilon > 0$ and $0 < \alpha < \beta \leq a$, and consider

$$\Gamma = \{\alpha + is, -\epsilon \leq s \leq \epsilon\} \cup \{\beta + is, -\epsilon \leq s \leq \epsilon\}$$
$$\cup \{r + i\epsilon, \alpha \leq r \leq \beta\} \cup \{r - i\epsilon, \alpha \leq r \leq \beta\},$$

then it is easy to see that

$$\int_\Gamma \int_0^a \left| \frac{1}{t-z} \right| d|z| < \infty.$$

Consequently, the last equality yields

$$\frac{1}{2\pi i} \int_\Gamma \int_0^a \frac{1}{t-z}dtdz < TXx, y > -\frac{\lambda^2}{2\pi i} \int_\Gamma \int_0^a \frac{1}{\lambda t - z}dtdz < XTx, y >$$
$$= \frac{1}{2\pi i} \int_\Gamma \int_0^a \frac{1}{t-z} < E([0,t])\sqrt{A}BXx, \sqrt{A}y > dtdz$$
$$-\frac{\lambda^2}{2\pi i} \int_\Gamma \int_0^a \frac{1}{\lambda t - z} < E([0,t])\sqrt{A}Bx, \sqrt{A}X^*y > dt.$$

Denote by $\Delta = Ins(\Gamma)$ the bounded connected component of Γ. Then, using Fubini's theorem, and in view of the arbitrary choice of ϵ, it follows that

$$\int_0^a \mathbf{1}_\Delta(t)dt < TXx, y >= \int_0^a \mathbf{1}_\Delta(t) < E([0,t])\sqrt{A}BXx, \sqrt{A}y > dt,$$

or

$$\int_\alpha^\beta \Big(< TXx, y > - < \sqrt{A}E([0,t])\sqrt{A}BXx, y >\Big)\, dt = 0,$$

thus

$$< TXx, y >=< \sqrt{A}E([0,t])\sqrt{A}BXx, y >, \; for\ a.\ e.\ t \in]0, a].$$

Due to the separability of $\mathcal{H}$, we have

$$TX = \sqrt{A}E([0,t])\sqrt{A}BX.$$

Let $(t_n)_{n\geq 0} \subset]0, a]$ verifying the last equation for any $n \geq 0$, and such that $t_n \searrow 0$, we get

$$TX = \sqrt{A}E(\{0\})\sqrt{A}BX = \sqrt{A}P_{ker(R)}\sqrt{A}BX.$$

Finally, note that $ker(R) = \{0\}$. Indeed, let $x \in \mathcal{H}$ be such that $Rx = 0$, then $\sqrt{A}B\sqrt{A}x = 0$, or $T\sqrt{A} = 0$, thus $\sqrt{A}x = 0$. Consequently $T^*x = BAx = 0$ which implies $x = 0$.
It follows that $TX = 0$ or equivalently, $X = 0$.

(2) Let $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$, and let $X \in \mathcal{L}(\mathcal{H})$ satisfying the equation $TX = \lambda XT$, then we similarly obtain

$$\begin{aligned} &\left(\int_0^a \mathbf{1}_\Delta(t)dt - \lambda \int_0^a \mathbf{1}_\Delta(\lambda t)dt\right) < TXx, y > \\ &= \int_{-a}^a \mathbf{1}_\Delta(t) < E(] - \infty, t])\sqrt{A}BXx, \sqrt{A}y > dt \\ &-\lambda^2 \int_{-a}^a \mathbf{1}_\Delta(\lambda t) < E(] - \infty, t])\sqrt{A}Bx, \sqrt{A}X^*y > dt. \end{aligned} \tag{2.1}$$

Thus, as did the analogous one in the proof of (1), and in view of the arbitrary choice of ϵ we get that $X = 0$. □

The following result characterizes the relation between the set of extended eigenvectors of the operator T given in Theorem 2.2, and the spectral measure associated to the self-adjoint operator R.

Theorem 2.3. *Let $A, B \in \mathcal{L}(\mathcal{H})$, and let $T = AB$ be such that T and T^* are injective. Consider $R = \sqrt{A}B\sqrt{A}$ and $a > \|R\|$, and let E denotes the spectral measure associated to R. Also, let $\lambda \in \mathbb{R}^*$, and let $X \in \mathcal{L}(\mathcal{H})\backslash\{0\}$. Then*

1. *If $\lambda \in]0, 1[$, then $TX = \lambda XT$ if and only if $\sqrt{A}E(]-\infty, t])\sqrt{A}BX = \lambda X\sqrt{A}E(]-\infty, t/\lambda])\sqrt{A}B$ for all $t \in]-\lambda a, \lambda a[$, $E(]-\infty, t])\sqrt{A}BX = 0$ for all $t \in [-a, -\lambda a]$ and $\sqrt{A}BX = E(] - \infty, t])\sqrt{A}BX$ for all $t \in [\lambda a, a]$.*
2. *If $\lambda \in] - 1, 0[$, then $\sqrt{A}E(]t, a])\sqrt{A}BX = \lambda X\sqrt{A}E(]-\infty, t/\lambda])\sqrt{A}B$ for all $t \in]\lambda a, -\lambda a[$, $E(] - \infty, t])\sqrt{A}BX = 0$ for all $t \in [-a, \lambda a]$ and $E(]t, a])\sqrt{A}BX$ for all $t \in [-\lambda a, a]$.*

(3) *If* $\lambda \in [1,+\infty[$, *then* $\sqrt{A}E(]-\infty,t])\sqrt{A}BX = \lambda X\sqrt{A}E(]-\infty,t/\lambda])$ $\sqrt{A}B$ *for all* $t \in]-a,a[$, $X\sqrt{A}E(]-\infty,t/\lambda]) = 0$ *for all* $t \in [-\lambda a,-a]$ *and* $TX = \lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B$ *for all* $t \in [a,\lambda a]$.

(4) *If* $\lambda \in]-\infty,-1]$, *then* $\sqrt{A}E(]t,a])\sqrt{A}BX = \lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B$ *for all* $t \in]-a,a[$, $X\sqrt{A}E(]-\infty,t/\lambda]) = 0$ *for all* $t \in [a,-\lambda a]$ *and* $TX = \lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B$ *for all* $t \in [\lambda a,-a]$.

Proof. Let $x,y \in \mathcal{H}$. We will show the statements (1) and (4). The other ones can be shown analogously. To do so, we will use the formula (2.1).

(1) Suppose that $TX = \lambda XT$, and let $\alpha,\beta \in [0,\lambda a]$ be such that $0 < \alpha < \beta \le \lambda a$, then

$$\left(\int_\alpha^\beta dt - \lambda\int_{\alpha/\lambda}^{\beta/\lambda} dt\right) < TXx,y >$$

$$= \int_\alpha^\beta < E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y > dt$$

$$-\lambda^2\int_{\alpha/\lambda}^{\beta/\lambda} < E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y > dt.$$

The separability of $\mathcal{H}$ yields

$$\sqrt{A}E(]-\infty,t])\sqrt{A}BX = \lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B,$$

for all $t \in [0,\lambda a]$.
If $\alpha,\beta \in [-\lambda a,0]$ such that $-\lambda a < \alpha < \beta \le 0$, then

$$0 = \int_\alpha^\beta < E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y > dt$$

$$-\lambda^2\int_{\alpha/\lambda}^{\beta/\lambda} < E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y > dt,$$

we thus obtain the same last equality for all $t \in [-\lambda a,0]$.
Now let $\alpha,\beta \in [-a,-\lambda a]$ be such that $-a < \alpha < \beta \le -\lambda a$, then

$$0 = \int_\alpha^\beta < E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y > dt.$$

Consequently, and since T is injective, we get

$$E(]-\infty,t])\sqrt{A}BX = 0,$$

for all $t \in [-a,-\lambda a]$.
Finally, if $\alpha,\beta \in [\lambda a,a]$ such that $\lambda a < \alpha < \beta \le a$, then

$$\int_\alpha^\beta dt < TXx,y >= \int_\alpha^\beta < E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y > dt$$

thus

$$TX = \sqrt{A}E(]-\infty,t])\sqrt{A}BX,$$

which gives the result for all $t \in [\lambda a,a]$.
For the other direction, we have

$$< TXx,y >=< \sqrt{A}E(]-a,a])\sqrt{A}BXx,y >,$$

by hypotheses,

$$\sqrt{A}E(]-a,-\lambda a])\sqrt{A}BX=\sqrt{A}E(]\lambda a,a])\sqrt{A}BX=0,$$

thus

$$<TXx,y>=<\sqrt{A}E(]-\lambda a,\lambda a])\sqrt{A}BXx,y>=\int_{-\lambda a}^{\lambda a}dE_{\sqrt{A}BXx,\sqrt{A}y}(t).$$

Consider the function

$$\begin{array}{rcccc}\varphi & : & [-a,a] & \to & [-\lambda a,\lambda a]\\ & & t & \mapsto & \lambda t\end{array}$$

Then, using the well-known formula of the image measure, we get

$$\begin{aligned}<TXx,y>&=\lambda\int_{-a}^{a}dE_{\sqrt{A}Bx,\sqrt{A}X^*y}(t)\\ &=\lambda<\sqrt{A}Bx,\sqrt{A}X^*y>=<\lambda XTx,y>,\end{aligned}$$

which implies that $TX=\lambda XT$.

(4) Suppose that $TX=\lambda XT$, and let $\alpha,\beta\in[0,a]$ be such that $0<\alpha<\beta\leq a$, then

$$\begin{aligned}\int_{\alpha}^{\beta}dt<TXx,y>&=\int_{\alpha}^{\beta}<E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y>dt\\ &-\lambda^2\int_{\beta/\lambda}^{\alpha/\lambda}<E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y>dt.\end{aligned}$$

The separability of $\mathcal{H}$ yields

$$TX=\sqrt{A}E(]-\infty,t])\sqrt{A}BX+\lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B,$$

for all $t\in[0,a]$. Equivalently

$$\sqrt{A}E(]t,a])\sqrt{A}BX=\lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B,\ \forall t\in[0,a].$$

If $\alpha,\beta\in[-a,0]$ such that $-a<\alpha<\beta\leq 0$, then

$$\begin{aligned}-\lambda\int_{\beta/\lambda}^{\alpha/\lambda}<TXx,y>&=\int_{\alpha}^{\beta}<E(]-\infty,t])\sqrt{A}BXx,\sqrt{A}y>dt\\ &-\lambda^2\int_{\beta/\lambda}^{\alpha/\lambda}<E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y>dt,\end{aligned}$$

we thus obtain the same last equality for all $t\in[-a,0]$.

Now let $\alpha,\beta\in[\lambda a,-a]$ be such that $\lambda a<\alpha<\beta\leq -a$, then

$$-\lambda\int_{\beta/\lambda}^{\alpha/\lambda}<TXx,y>=-\lambda^2\int_{\beta/\lambda}^{\alpha/\lambda}<E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y>dt,$$

Consequently

$$TX=\lambda X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B,$$

for all $t\in[\lambda a,-a]$.

Finally, if $\alpha,\beta\in[a,-\lambda a]$ such that $a<\alpha<\beta\leq-\lambda a$, then

$$0=-\lambda^2\int_{\beta/\lambda}^{\alpha/\lambda}<E(]-\infty,t])\sqrt{A}Bx,\sqrt{A}X^*y>dt,$$

thus

$$X\sqrt{A}E(]-\infty,t/\lambda])\sqrt{A}B=0,$$

and the dense image of all A and B yields the result for all $t\in[a,-\lambda a]$.
For the other direction, we have

$$TX=\sqrt{A}E(]-a,a])\sqrt{A}BX,$$

by hypotheses,

$$\sqrt{A}E(]-a,a])\sqrt{A}BX=\lambda X\sqrt{A}E(]-\infty,a/\lambda])\sqrt{A}B-\lambda X\sqrt{A}E(]-\infty,-a/\lambda])\sqrt{A}B.$$

thus

$$TX=\lambda X\sqrt{A}E(]a/\lambda,-a/\lambda])\sqrt{A}B.$$

But

$$X\sqrt{A}E(]-a/\lambda,a])\sqrt{A}B=X\sqrt{A}E(]-a,a/\lambda])\sqrt{A}B=0,$$

which means that

$$TX=\lambda X\sqrt{A}E(]-a,a])\sqrt{A}B=\lambda XT,$$

and the theorem is proved. □

Corollary 2.4. *Let $T\in\mathcal{L}(\mathcal{H})$ be an injective self-adjoint operator, let $a>\|T\|$ and let E denotes the spectral measure associated to T. Also, let $\lambda\in\mathbb{R}^*$, and let $X\in\mathcal{L}(\mathcal{H})$ satisfying the equation $TX=\lambda XT$. Then*

(1) *If $\lambda\in]0,1[$, then $E(]-\infty,t])X=XE(]-\infty,t/\lambda])$ for all $t\in]-\lambda a,\lambda a[$, $E(]-\infty,t])X=0$ for all $t\in[-a,-\lambda a]$ and $E(]-\infty,t])X=X$ for all $t\in[\lambda a,a]$.*
(2) *If $\lambda\in]-1,0[$, then $X=E(]-\infty,t])X-XE(]-\infty,t/\lambda])$ for all $t\in]\lambda a,-\lambda a[$, $E(]-\infty,t])X=0$ for all $t\in[-a,\lambda a]$ and $E(]-\infty,t])X=X$ for all $t\in[-\lambda a,a]$.*
(3) *If $\lambda\in]1,+\infty[$, then $E(]-\infty,t])X=XE(]-\infty,t/\lambda])$ for all $t\in]-a,a[$, $XE(]-\infty,t/\lambda])=0$ for all $t\in[-\lambda a,-a]$ and $XE(]-\infty,t/\lambda])=X$ for all $t\in[a,\lambda a]$.*
(4) *If $\lambda\in]-\infty,-1[$, then $X=E(]-\infty,t])X-XE(]-\infty,t/\lambda])$ for all $t\in]-a,a[$, $XE(]-\infty,t/\lambda])=0$ for all $t\in[a,-\lambda a]$ and $XE(]-\infty,t/\lambda])=-X$ for all $t\in[\lambda a,-a]$.*

Proof. We show one of theses statements, the other ones can be shown analogously. To do so, we will use the formula

$$\left(\int_0^a \mathbf{1}_\Delta(t)dt-\lambda\int_0^a \mathbf{1}_\Delta(\lambda t)dt\right)<Xx,y>$$

$$=\int_{-a}^a \mathbf{1}_\Delta(t)<E(]-\infty,t])Xx,y>dt-\lambda\int_{-a}^a \mathbf{1}_\Delta(\lambda t)<XE(]-\infty,t])x,y>dt,$$

obtained by using (2.1) when $B=T$ and $A=I$. Now suppose that $\lambda\in]0,1[$ and $0<\lambda a<\alpha<\beta\le a$, then

$$\int_\alpha^\beta <Xx,y>dt=\int_\alpha^\beta <E(]-\infty,t])Xx,y>dt,$$

or

$$<Xx,y>=<E(]-\infty,t])Xx,y>,\ \forall t\in[\lambda a,a].$$

Since $\mathcal{H}$ is separable, we get $X=E(]-\infty,t])X$ as desired. □

3. Compact Self-adjoint Operators

The most general case of this class of operators will be described in the following lemma of spectral decomposition. The other cases can be shown similarly.

Lemma 3.1. *Let T be a compact self-adjoint operator such that $\sigma_p(T) = (\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \neq 0$ for all $n \in \mathbb{N}$, and $\lambda_i \neq \lambda_j$ for all $i \neq j$. If we denote, for all $n \in \mathbb{N}$, by $\mathcal{M}(\lambda_n)$ the (finite) multiplicity of the eigenvalue λ_n, and by $(e_{n,j})_{j=1}^{\mathcal{M}(\lambda_n)}$ the orthonormal basis of the eigenspace associated with λ_n. then the set $\{e_{n,j} : n \in \mathbb{N}, j = 1, ..., \mathcal{M}(\lambda_n)\}$ forms an orthonormal basis of $\mathcal{H}$.*

Now we can show

Theorem 3.2. *Suppose that T verifies the same hypotheses of the last lemma. Then $\sigma_{ext}(T) = \{\frac{\lambda_i}{\lambda_j} : i, j \in \mathbb{N}\}$ and for any $i, j \in \mathbb{N}$, we have*

$$E_{ext}(\frac{\lambda_i}{\lambda_j}) = \mathrm{span}\{\sum_{k=1}^{\mathcal{M}(\lambda_m)}\sum_{l=1}^{\mathcal{M}(\lambda_n)} c_{kl}e_{m,k} \otimes e_{n,l}\ \forall m, n \in \mathbb{N} \text{ where } \frac{\lambda_m}{\lambda_n} = \frac{\lambda_i}{\lambda_j}\}.$$

Proof. Let $\lambda \in \mathbb{C}$ and $X \in \mathcal{L}(\mathcal{H})$ be such that

$$TX = \lambda XT,$$

then by Lemma for all $j \in \mathbb{N}$ we have

$$TXe_{j,l} = \lambda\lambda_j Xe_{j,l},\ l = 1, ..., \mathcal{M}(\lambda_n).$$

If $X \neq 0$, then necessarily there are $i, j \in \mathbb{N}$, such that

$$\lambda = \frac{\lambda_i}{\lambda_j} \text{ and } Xe_{j,l} = \sum_{k=1}^{\mathcal{M}(\lambda_i)} c_{kl}e_{i,k}\ l = 1, ..., \mathcal{M}(\lambda_j),$$

and at least one of constants c_{kl} is nonzero. consequently

$$TX = \frac{\lambda_i}{\lambda_j}XT.$$

By applying on $e_{n,l}$ for all $n \in \mathbb{N}$ and $l = 1, ..., \mathcal{M}(\lambda_n)$, we obtain

$$Xe_{n,l} = \begin{cases} \sum_{k=1}^{\mathcal{M}(\lambda_m)} c_{kl}e_{m,k} & \text{if } \frac{\lambda_m}{\lambda_n} = \frac{\lambda_i}{\lambda_j} \\ 0 & \text{otherwise,} \end{cases}$$

as desired. □

Remark 3.3. *We could also obtain this result by using Corollary 2.4.*

4. Self-adjoint Operators Without Point Spectrum

In this section we will consider another class of self-adjoint operators; those which have no point spectrum, and zero is a point in the continuous spectrum. When the spectral multiplicity is constant, the description of the extended eigenspaces is simpler and can be obtained directly by using Lemma 1.3. For simplicity, we consider a particular case where the multiplicity is equal to one. Let $T \in \mathcal{L}(L^2[0, 1])$ defined by

$$Tf(x) = xf(x),\ \forall f \in L^2[0, 1]\ \forall x \in [0, 1], \tag{4.1}$$

then it is easy to show that $\sigma(T) = \sigma_c(T) = [0,1]$. Let now $\varphi : [0,1] \to [0,1]$ be a measurable function. A composition operator C_φ is defined as $C_\varphi f(x) = f(\varphi(x))$. The composition operator $C_{\lambda x}$, will be denoted simply as C_λ. Also, if $\varphi \in L^\infty[0,1]$, we define the operator of multiplication by φ as $M_\varphi f(x) = \varphi(x)f(x)$.
Now we have the following theorem

Theorem 4.1. *Let T be defined by (4.1), then $\sigma_{ext}(T) =]0,\infty[$, and if we suppose that $X \in \mathcal{L}(L^2[0,1])$ is a nonzero operator. Then :*

(1) *if $\lambda \in]0,1]$, then $TX = \lambda XT$ if and only if $XC_\lambda = M_{X1}$;*
(2) *if $\lambda \in]1,\infty[$, then $TX = \lambda XT$ if and only if $X = M_{X1}C_{1/\lambda}$.*

Proof. First note that T is injective, there is no nonzero solution of the equation $TX = 0$, and henceforth for the rest of this proof we suppose that $\lambda \in \mathbb{C}^*$.
Let X be a bounded operator on $L^2[0,1]$ satisfying the equation $TX = \lambda XT$. Then for all $f \in L^2[0,1]$ and for any $n \in \mathbb{N}$

$$T^n Xf = \lambda^n XT^n f.$$

In particular,

$$T^n X\mathbf{1} = \lambda^n XT^n \mathbf{1},$$

that is,

$$Xt^n = (\frac{t}{\lambda})^n X\mathbf{1},\ \forall\ n \in \mathbb{N},$$

hence

$$Xp(t) = p(\frac{t}{\lambda})X\mathbf{1},\ \forall\ p \in \mathbb{C}[X].$$

Now we suppose that $\lambda \in \mathbb{C}\backslash[0,\infty[$ and we consider $K = [0,1] \cup [0,\frac{1}{\lambda}]$. Let $f \in L^2[0,1]$ then we can see f as element of $L^2(K)$ by considering $f_{|]0,\lambda]} = 0$, *a. e.* According to Lemma 1.3, there is a sequence $(p_n)_{n\in\mathbb{N}} \subset \mathbb{C}[X]$ that converges to f in $L^2(K)$. That is,

$$Xf(t) = \lim_{n\to\infty} Xp_n(t) = \lim_{n\to\infty} p_n(\frac{t}{\lambda})X\mathbf{1} = 0,\ for\ a.\ e.\ t \in [0,1].$$

Thus $X = 0$. For (1), we obtained that $Xp(\lambda t) = p(t)X\mathbf{1}$ for all $p \in \mathbb{C}[X]$. Since the set of polynomials is dense in $L^2[0,1]$, $Xf(\lambda t) = f(t)X\mathbf{1}$ for all $f \in L^2[0,1]$, which means that $XC_\lambda = M_{X1}$. It is easy to see that if $XC_\lambda = M_{X1}$, then $TX = \lambda XT$. We show (2) with the same arguments as in the proof of (1).

□

Remark 4.2. *The case of arbitrary constant spectral multiplicity will be studied in the final version of this article.*

Corollary 4.3. *Let $\varphi : [0,1] \to [0,1]$ be a measurable function. $TC_\varphi = \lambda C_\varphi T$ if and only if $\lambda \geq 1$ and $\varphi(x) = x/\lambda$, $x \in [0,1]$.*

Proof. It is obvious that $C_\varphi\mathbf{1} = \mathbf{1}$. If $\lambda \in]0,1[$ then according to part (1) of last theorem, $TC_\varphi = \lambda C_\varphi T$ if and only if $C_\varphi C_\lambda = I$ which contradicts the fact that $C_{1/\lambda}$ is not left-invertible. Suppose now that $\lambda \geq 1$, then using same theorem, $TC_\varphi = \lambda C_\varphi T$ if and only if $C_\varphi = C_{1/\lambda}$, or $\varphi(x) = x/\lambda$. This completes the proof.

□

References

[1] H. Alkanjo. On extended eigenvalues and extended eigenvectors of truncated shift. *Preprint.*

[2] A. Biswas, A. Lambert, and S. Petrovic. Extended eigenvalues and the Volterra operator. *Glasg. Math. J.*, 44(3):521–534, 2002.

[3] A. Biswas and S. Petrovic. On extended eigenvalues of operators. *Integral Equations Operator Theory*, 55(2):233–248, 2006.

[4] M. T. Karaev. On extended eigenvalues and extended eigenvectors of some operator classes. *Proc. Amer. Math. Soc.*, 134(8):2383–2392 (electronic), 2006.

[5] M. Rosenblum. On the operator equation $BX - XA = Q$. *Duke Math. J.*, 23:263–269, 1956.

Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne

E-mail address: cassier@math.univ-lyon1.fr

E-mail address: alkanjo@math.univ-lyon1.fr